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Almost Periodic Solutions of Linear Partial Differential Equations*

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Let L be an arbitrary linear partial differential operator and let f be an almost periodic function for t in R^m. In this paper we present sufficient conditions that a bounded solution u of $Lu = f$ be almost periodic. Our work generalizes the theorem of Sibuya [5] for Poisson's equation and the theorems of Favard [3] and Bochner [1] for ordinary differential equations.

1. INTRODUCTION

In this paper we will study the question of existence of almost periodic solutions of the system of linear partial differential equations

$$\sum_{j=1}^{n} L_{ij} u_j = f_i , \qquad 1 \leqslant i \leqslant n, \tag{1}$$

on R^m, where L_i is an arbitrary linear partial differential operator on R^m given by

$$L_{ij} = \sum_{\alpha} a_{\alpha ij} D^{\alpha}$$

and the summation is finite. (We use the standard notation for partial differential operators (cf. [2]). It will be more convenient to write the system (1) in the form

$$Lu = f, \tag{2}$$

where u and f are now viewed as mappings of R^m into R^n. The order k of L is defined to be the maximum of the orders of the L_i, i.e., k denotes the highest order derivative appearing in Eq. (1).

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We will assume that the coefficients $a_{\alpha ij}$ and f_i are continuous and almost periodic functions of $t = (t^1,..., t^m)$ in R^m. Recall that g is an *almost periodic* function of t in R^m if for every sequence $\beta' = \{\beta_n'\}$ in R^m, there is a subsequence $\beta = \{\beta_n\}$ such that $\lim g(t + \beta_n)$ converges uniformly for t in R^m. This notion of almost periodicity, which is due to Bochner for the case $R^m = R^1$, is equivalent to the Bohr concept of almost periodicity, which is defined in terms of a relatively dense set of translation numbers.

We define the *hull* $H(L, f)$ to be the collection of all linear partial differential equations $L^*u = f^*$ where the coefficients $a^*_{\alpha ij}$ and f_i^* are related to $a_{\alpha ij}$ and f_i by

$$\lim a_{\alpha ij}(t + \beta_n) = a^*_{\alpha ij}(t) \qquad \text{and} \qquad \lim f_i(t + \beta_n) = f_i^*(t), \qquad t \in R^m, \quad (3)$$

for some sequence $\beta = \{\beta_n\}$, which is independent of α, i, and j. Since $a_{\alpha ij}$ is assumed to be almost periodic the limit in Eqs. (3) is in fact uniform for all t in R^m, although we shall only require pointwise convergence in Eqs. (3) to define the hull. $H(L, 0)$ will denote the hull of the homogeneous equation $Lu = 0$.

A solution u of Eq. (2) is said to be *C^k-bounded* if $u = (u_1 ,..., u_n)$ together with all derivatives up to and including order k are bounded and uniformly continuous on R^m.

Our first theorem is concerned with classical solutions of Eq. (2).

THEOREM 1. *Let L be a linear partial differential operator of order k and assume that the coefficients $a_{\alpha ij}$ and f_i are almost periodic functions of t in R^m. Assume further that every C^k-bounded solution of every homogeneous equation $L^*v = 0$ in the hull $H(L, 0)$ is almost periodic. If u is a C^k-bounded solution of any equation $L^*u = f^*$ in the hull $H(L, f)$, then u is almost periodic.*

This theorem generalizes to partial differential equations similar results of Favard [3] and Bochner [1] for ordinary differential equations. The proof of this theorem, which we present in Section 3, follows essentially the argument of Bochner, with appropriate modifications for partial differential equations.

The concept of C^k-boundedness requires that the highest order derivative be uniformly continuous as well as bounded. In certain special cases one can prove that boundedness implies uniform continuity. For example, this is true for ordinary differential equations. It is also true for Poisson's equation $\Delta u = f$, provided one uses the concept of a "solution in the sense of distribution" (cf. [2, 5]).

The second problem we study in this paper is the question of extending Theorem 1 to include the study of generalized solutions or, as they are also called, solutions in the sense of distribution. In particular we study the behavior of bounded solutions of weakly coupled systems of the form

$$\Delta u_i + \sum_{j=1}^{n} a_{ij} u_j = f_i, \qquad 1 \leqslant i \leqslant n, \tag{4}$$

where Δ is the Laplacian operator and the coefficients a_{ij} and f_i are almost periodic functions of t in R^m. Once again it is more convenient to write Eq. (4) in the form

$$\Delta u + Au = f. \tag{5}$$

The *hull* of Eq. (5) is defined to be the collection $H(A, f)$ of all partial differential equations

$$\Delta u + A^* u = f^*,$$

where the coefficients A^* and f^* are related to A and f by

$$A^*(t) = \lim A(t + \beta_n) \qquad \text{and} \qquad f^*(t) = \lim f(t + \beta_n), \quad t \in R^m,$$

for some sequence $\beta = \{\beta_n\}$ in R^m. For the homogeneous equation $\Delta v + Av = 0$, the hull is simply $H(A, 0)$.

We define a generalized solution of Eq. (5) to be a continuous function $u\colon R^m \to R^n$ with the property that for every C^∞-function $\phi : R^m \to R^n$ with compact support one has

$$\int_{R^m} [(u, \Delta\phi) + (Au, \phi)]\, dt = \int_{R^m} (f, \phi)\, dt.$$

We use (u, u) to denote an inner product on R^n and $|u| = (u, u)^{1/2}$ the induced norm.

For Eq. (5) we can prove the following result

THEOREM 2. (A) *Let A and f be almost periodic functions of t in R^m. Assume that if v is any nontrivial bounded generalized solution of any homogeneous equation $\Delta v + A^* v = 0$ in the hull $H(A, 0)$, then*

$$\inf\{|v(t)| : t \in R^m\} > 0.$$

If there exists a bounded generalized solution u of Eq. (5), then there exists an almost periodic generalized solution $\tilde{u}$ of Eq. (5).

(B) *If every bounded generalized solution of every homogeneous equation in the hull $H(A, 0)$ is almost periodic, then every bounded generalized solution of Eq. (5) is almost periodic.*

Theorem 2 includes as a special case the theorem of Sibuya for Poisson's equation, which one gets by setting $n = 1$ and $A = 0$ (cf. [5]). In this case, the only bounded generalized solutions of the homogeneous equation $\Delta u = 0$ are constant functions.

Also if one sets $m = 1$ and replaces Δ with the ordinary differential operator D, where $Du_i = du_i/dt$, then our theorem includes the well-known result of Favard [3, pp. 90–94] for ordinary differential equations.

The proof of Theorem 2 is given in Section 4.

2. Almost Periodic and Almost Automorphic Functions

The concept of almost periodicity can be reformulated in terms of certain translation operators T_β (cf. Bochner [1]). More precisely, let $\beta = \{\beta_n\}$ be any sequence in R^m and let $f: R^m \to R^n$ be any continuous function. If the limit

$$\lim f(t + \beta_n) = g(t) \tag{6}$$

exists for each t in R^m, then we define T_β by $g = T_\beta f$.

The next lemma gives the desired characterization of almost periodicity. It was proved originally for complex-valued functions of a real variable by Bochner [1], but the same argument applies in the more general case.

Lemma 1 (Bochner). *A continuous function $f: R^m \to R^n$ is almost periodic in t if and only if for any two sequences β' and γ', there exist subsequences β and γ such that*

$$T_{\beta+\gamma} f = T_\beta T_\gamma f \tag{7}$$

Note that Eq. (7) is equivalent to

$$\lim_{n\to\infty} f(t + \gamma_n + \beta_n) = \lim_{m\to\infty} \lim_{n\to\infty} f(t + \gamma_n + \beta_m), \qquad t \in R^m.$$

A continuous function $f: R^m \to R^n$ is said to be *almost automorphic* if for any sequence β' there is a subsequence β such that

$$T_{-\beta} T_\beta f = f. \tag{8}$$

It is clear that every almost periodic function is almost automorphic.

Before we turn to the proof of our theorems, let us comment briefly on behavior of the limit in Eq. (6). This limit is only assumed to exist pointwise. However, the only time we shall apply the translation operator T_β will be to functions f that are bounded and uniformly continuous. Therefore the limit in Eq. (6) will actually be uniform on compact sets in R^m.

3. Proof of Theorem 1

The proof of this theorem follows the argument of Bochner [1] used for studying ordinary differential-difference equations. The first step is to show that Theorem 1 remains valid if one replaces "almost periodic" with "almost automorphic." The concept of the hull in this case is the same as defined above for almost periodic functions.

Lemma 2. *Let L be a linear partial differential operator of order k and assume that the coefficients $a_{\alpha ij}$ and f_i are almost automorphic functions of t in R^m. Assume further that every C^k-bounded solution of every homogeneous equation $L^*v = 0$ is almost automorphic. If u is a C^k-bounded solution of any equation $L^*u = f^*$ in the hull, then u is almost automorphic.*

Proof. Let A_β denote the translation operator $T_{-\beta}T_\beta$. Then a continuous function $g: R^m \to R^n$ is almost automorphic if and only if for every sequence β' there is a subsequence β such that $A_\beta g = g$.

Since the coefficients $a_{\alpha ij}$ and f_i are almost automorphic, for every sequence β' there is a subsequence β such that all the limits $T_\beta a_{\alpha ij}$ and $T_\beta f_i$ exist. If we let L_β denote the partial differential operator with coefficients $T_\beta a_{\alpha ij}$, then by picking a subsequence if necessary, $Lu = f$ implies that

$$T_\beta Lu = L_\beta(T_\beta u) = T_\beta f.$$

If we apply $T_{-\beta}$ to the last equation, and by picking a further subsequence if necessary, we get

$$A_\beta Lu = L(A_\beta u) = f, \tag{9}$$

since the coefficients are almost automorphic. By subtracting $Lu = f$ from Eq. (9) we get $L(A_\beta u - u) = 0$. It follows from the hypothesis that $A_\beta u - u$ is almost automorphic, hence we have $A_\beta u = u + g$, where g is almost automorphic. Now by applying A_β successively, and by taking further subsequences if necessary, we get $A_\beta{}^n u = u + ng$ or

$$g = n^{-1}(A_\beta{}^n u - u). \tag{10}$$

Clearly we have $\| A_\beta u \|_\infty \leqslant \| u \|_\infty$, where $\| \cdot \|_\infty$ denotes the supnorm. It then follows from Eq. (9) that $\| g \|_\infty \leqslant n^{-1} 2 \| u \|_\infty$. Since n is arbitrary we have $g = 0$. Hence $A_\beta u = u$. Q.E.D.

The next lemma gives another characterization of almost periodicity.

Lemma 3. *Let $g: R^m \to R^n$ be a continuous function. Then g is almost* *periodic in t if and only if for any two sequences β' and γ' there exist subsequences β and γ such that*

(i) *$T_\beta T_\gamma g$ is almost automorphic, and*

(ii) *$Bg = g$,*

where

$$B = T_{-(\beta+\gamma)} T_\beta T_\gamma .$$

Proof. If (i) and (ii) hold, then by applying $T_{(\beta+\gamma)}$ to $Bg = g$ and by taking a subsequence if necessary, we get

$$T_\beta T_\gamma g = T_{(\beta+\gamma)} T_{-(\beta+\gamma)} T_\beta T_\gamma g = T_{(\beta+\gamma)} g,$$

hence g is almost periodic.

Conversely if g is almost periodic then for any two sequences β' and γ' there exist subsequences β and γ such that the limits $T_\beta T_\gamma g$, $T_{(\beta+\gamma)} g$, and Bg exist. Furthermore each of these three functions is almost periodic, and therefore almost automorphic. Hence (i) holds and

$$Bg = T_{-(\beta+\gamma)} T_\beta T_\gamma g = T_{-(\beta+\gamma)} T_{(\beta+\gamma)} g = g,$$

i.e., (ii) holds. Q.E.D.

Proof of Theorem 1. We begin by following the argument of Lemma 2 but now using the operator B instead of A_β. Since the coefficients $a_{\alpha ij}$ and f_i are almost periodic if we apply the operator B to $Lu = f$ we get

$$BLu = L(Bu) = f. \tag{11}$$

More precisely, for any two sequences β' and γ' there exist subsequences β and γ such that Eq. (11) holds for $B = T_{-(\beta+\gamma)} T_\beta T_\gamma$. By subtracting $Lu = f$ from Eq. (11) we get $L(Bu - u) = 0$. By the hypothesis of Theorem 1, we see that $Bu = u + g$, where g is almost periodic. Now by applying B successively and using the same argument as used in Lemma 2 we conclude that $g = 0$. Hence $Bu = u$.

If we show that, for appropriate subsequences, the function $T_\beta T_\gamma u$ is almost automorphic, then it will follow from Lemma 3 that u is almost periodic. However, if we apply $T_\beta T_\gamma$ to $Lu = f$, and by taking a subsequence if necessary, we get

$$L_{\beta,\gamma}(T_\beta T_\gamma u) = T_\beta T_\gamma f,$$

where $L_{\beta,\gamma}$ is the operator with coefficients $T_\beta T_\gamma a_{\alpha ij}$. Lemma 3 assures us that $T_\beta T_\gamma f$ is almost automorphic, and Lemma 2 then implies that $T_\beta T_\gamma u$ is almost automorphic. Q.E.D.

Remark 1. The definition of C^k-boundedness can be relaxed somewhat. All we really need is that the derivatives $D^\alpha u_j$, which correspond to nonzero coefficients $a_{\alpha ij}$, be bounded and uniformly continuous in R^m.

Remark 2. The same methods can also be applied to partial differential-difference operators where L_{ij} is now of the form

$$L_{ij}u_j(t) = \sum_h \sum_\alpha a_{\alpha ijh} D^\alpha u_j(t + \sigma_h),$$

where both summations are finite. The corresponding theorem has exactly the same form as Theorem 1.

4. Proof of Theorem 2

In Theorem 1 we showed that the given C^k-bounded solution u is almost periodic. For Theorem 2(A), we seek an existence theorem. The almost periodic solution need not be the given bounded solution. We shall show that there exists a unique solution $\tilde{u}$ of $\Delta u + Au = f$ with minimum supnorm. We will then show that $\tilde{u}$ is almost periodic. Throughout this section we shall use the term "solution" to refer to the concept of "generalized solution" defined above.

Lemma 4. *Assume that the hypotheses of Theorem* 2(A) *are satisfied. If for some equation $\Delta u + A^*u = f^*$ in the hull $H(A, f)$ there exists an almost periodic solution u^*, then every equation in the hull has an almost periodic solution.*

Proof. If $\beta = \{\beta_n\}$ is a sequence in R^m such that the limits $T_\beta u^*$, $T_\beta A^*$ and $T_\beta f^*$ exist, then $T_\beta u^*$ is an almost periodic solution of

$$\Delta u + T_\beta A^* u = T_\beta f^*.$$

Finally as we vary β over all such sequences $(T_\beta A^*, T_\beta f^*)$ assumes every value in the hull $H(A, f)$. Q.E.D.

Lemma 5. *Let A and f be bounded on R^m and assume that u is a bounded solution of $\Delta u + Au = f$ with $\| u \|_\infty \leqslant B$. Then there is a constant K, depending on B, $\| A \|_\infty$ and $\| f \|_\infty$, but not on u, such that*

$$| u(t) - u(s) | \leqslant K \, | t - s | .$$

For a proof of Lemma 5 we refer to Courant [2, Chap. 4].

For each (A^*, f^*) in the hull $H(A, f)$ and for each real number B, $0 \leqslant B < \infty$, we define the solution set $\mathfrak{S}(A^*, f^*; B)$ to be the collection of all solutions u of $\Delta u + A^*u = f^*$ with $\| u \|_\infty \leqslant B$. We let

$$\mathfrak{S}(B) = \bigcup \mathfrak{S}(A^*, f^*; B),$$

where the union is taken over all (A^*, f^*) in the hull $H(A, f)$.

LEMMA 6. *Let A and f be bounded on R^m. Then for each B, the solution set $\mathfrak{S}(A, f; B)$ is a compact subset of $C(R^m, R^n)$ in the topology of uniform convergence on compact sets.*

Proof. The fact that the solution set $\mathfrak{S}(A, f; B)$ is relatively compact follows from the Arzela–Ascoli Theorem and Lemma 4. Furthermore, it is easily verified that $\mathfrak{S}(A, f; B)$ is closed, hence it is compact. Q.E.D.

LEMMA 7. *Assume that the hypotheses of Theorem* 2(A) *are satisfied. Then for each B, the solution set $\mathfrak{S}(B)$ is a compact subset of $C(R^m, R^n)$ is the topology of uniform convergence on compact sets.*

Proof. Since the hypotheses of Theorem 2(A) are satisfied the norms $\| A^* \|_\infty$ and $\| f^* \|_\infty$ are bounded as (A^*, f^*) vary over the hull $H(A, f)$. It then follows from Lemma 4 and the Arzela–Ascoli Theorem that $\mathfrak{S}(B)$ is relatively compact. However, if $\{u_n\}$ is a sequence in $\mathfrak{S}(B)$ with

$$\int_{R^m} [(u_n, \Delta\phi) + (A_n u_n, \phi)]\, dt = \int_{R^m} (f_n, \phi)\, dt,$$

where $(A_n, f_n) \in H(A, f)$ and $u_n \to u^*$, then by choosing a suitable subsequence if necessary, we have $(A_n, f_n) \to (A^*, f^*)$ in $H(A, f)$. So by the dominated convergence theorem we get

$$\int_{R^m} [(u^*, \Delta\phi) + (A^* u^*, \phi)]\, dt = \int_{R^m} (f^*, \phi)\, dt,$$

that is, $u^* \in \mathfrak{S}(A^*, f^*; B)$. Hence $\mathfrak{S}(B)$ is closed and therefore compact. Q.E.D.

The solution set $\mathfrak{S}(A, f; B)$ may be empty. However, the hypotheses of Theorem 2(A) imply that for some B, $\mathfrak{S}(A, f; B)$ is nonempty.

LEMMA 8. *Assume that the hypotheses of Theorem* 2(A) *are satisfied and let B be any real number with the property that $\mathfrak{S}(A, f; B)$ is nonempty. Then for every (A^*, f^*) in the hull $H(A, f)$, the solution set $\mathfrak{S}(A^*, f^*; B)$ is nonempty.*

Proof. Let $u \in \mathfrak{S}(A, f; B)$ and let $(A^*, f^*) \in H(A, f)$. Now choose a sequence β in R^m so that $T_\beta A = A^*$, $T_\beta f = f^*$ and so that the limit $T_\beta u$ exists. It follows then that $T_\beta u$ is a solution of $\Delta u + A^*u = f^*$. Since $\| T_\beta u \|_\infty \leqslant \| u \|_\infty$, we see that $T_\beta u \in \mathfrak{S}(A^*, f^*; B)$. Q.E.D.

For the rest of the argument we shall fix B, so that the solution sets $\mathfrak{S}(A^*, f^*; B)$ are nonempty.

For each (A^*, f^*) in the hull $H(A, f)$ we define

$$\nu(A^*, f^*) = \inf\{\| u \|_\infty : u \in \mathfrak{S}(A^*, f^*; B)\}.$$

(It is easy to verify that ν does not depend on the choice of B we made above.) We shall say that a solution $\tilde{u}$ of $\Delta u + A^*u = f^*$ *optimal* if $\| \tilde{u} \|_\infty \leqslant \nu(A^*, f^*)$. We can now prove the following interesting fact about optimal solutions.

LEMMA 9. *Assume that the hypotheses of Theorem* 2(A) *are satisfied. Then for each* (A^*, f^*) *in the hull* $H(A, f)$ *there is precisely one optimal solution* $\tilde{u}$ *of* $\Delta u + A^*u = f^*$.

Proof. First we shall establish existence. Let $\{u_n\}$ be a sequence of solutions in $\mathfrak{S}(A^*, f^*; B)$ with $\| u_n \|_\infty \to \nu(A^*, f^*)$. It follows from the compactness of $\mathfrak{S}(A^*, f^*; B)$ that we can also assume that $\{u_n\}$ converges uniformly on compact sets in R^m to a solution $\tilde{u}$ in $\mathfrak{S}(A^*, f^*; B)$. Finally it is clear that $\| \tilde{u} \|_\infty \leqslant \lim \| u_n \|_\infty = \nu(A^*, f^*)$, hence $\tilde{u}$ is an optimal solution.

Now we shall prove uniqueness. Let $\tilde{u}_1$ and $\tilde{u}_2$ be two optimal solutions of $\Delta u + A^*u = f^*$. Then $u = (\tilde{u}_1 + \tilde{u}_2)/2$ is also a solution of $\Delta u + A^*u = f^*$ and $v = (\tilde{u}_1 - \tilde{u}_2)/2$ is a solution of the homogeneous equation $\Delta u + A^*u = 0$. If $\tilde{u}_1 \neq \tilde{u}_2$, then there is an $\alpha > 0$ such that

$$\inf\{| v(t) | : t \in R^m\} \geqslant \alpha,$$

by the hypotheses of Theorem 2. On the other hand, in terms of the inner product on R^n, we get

$$| u(t) |^2 + | v(t) |^2 = \tfrac{1}{2}(\tilde{u}_1(t), \tilde{u}_1(t)) + \tfrac{1}{2}(\tilde{u}_2(t), \tilde{u}_2(t)) \leqslant \nu(A^*, f^*)^2.$$

Hence

$$| u(t) |^2 \leqslant \nu(A^*, f^*)^2 - | v(t) |^2 \leqslant \nu(A^*, f^*)^2 - \alpha^2,$$

or $\| u \|_\infty < \nu(A^*, f^*)$, which contradicts the fact that $\nu(A^*, f^*) \leqslant \| u \|_\infty$. Hence $\tilde{u}_1 = \tilde{u}_2$. Q.E.D.

We will want to show that if we apply a shift operator T_β to an optimal solution $\tilde{u}$, that this will map $\tilde{u}$ onto another optimal solution. For this we will need the following result.

LEMMA 10. *Assume that the hypotheses of Theorem* 2(A) *are satisfied. Define* ν_0 *by*

$$\nu_0 = \inf\{\nu(A^*, f^*): (A^*, f^*) \in H(A, f)\}.$$

Then for all (A^*, f^*) *in the hull* $H(A, f)$ *one has* $\nu(A^*, f^*) = \nu_0$.

Proof. First we shall show that there is an (A_0, f_0) in the hull $H(A, f)$ such that $\nu_0 = \nu(A_0, f_0)$. Pick a sequence $\{(A_n, f_n)\}$ in the hull such that $\nu(A_n, f_n) \to \nu_0$. Let $\{\tilde{u}_n\}$ denote the corresponding optimal solutions. Since $\{\tilde{u}_n\}$ belong to the compact set $\mathfrak{S}(B)$ and since $H(A, f)$ is compact, we can find convergent subsequences, say that $\tilde{u}_n \to \tilde{u}_0$, $A_n \to A_0$, and $f_n \to f_0$. It follows from the argument of Lemma 7 that $\tilde{u}_0$ is a solution of $\Delta u + A_0 u = f_0$. Furthermore, one has

$$\nu(A_0, f_0) \leqslant \| \tilde{u}_0 \|_\infty \leqslant \lim \| \tilde{u}_n \|_\infty = \lim \nu(A_n, f_n) = \nu_0 \leqslant \nu(A_0, f_0).$$

Now by applying Lemma 8 with $B = \nu_0$ we see that for all (A^*, f^*) in the hull $H(A, f)$, the solution set $\mathfrak{S}(A^*, f^*; \nu_0)$ is nonempty. This implies that $\nu(A^*, f^*) \leqslant \nu_0$. Since one always has $\nu_0 \leqslant \nu(A^*, f^*)$, we get $\nu_0 = \nu(A^*, f^*)$. Q.E.D.

LEMMA 11. *Assume that the hypotheses of Theorem* 2(A) *are satisfied and let* $\tilde{u}$ *be the unique optimal solution of* $\Delta u + Au = f$. *Then* $\tilde{u}$ *is almost periodic on* R^m.

Proof. We shall apply Lemma 1. Let β' and γ' be arbitrary sequences in R^m. Now choose subsequences β and γ so that the following limits exist:

$T_\gamma \tilde{u}$, $T_\gamma A$, $T_\gamma f$, $T_\beta T_\gamma \tilde{u}$, $T_\beta T_\gamma A$, $T_\beta T_\gamma f$, $T_{\beta+\gamma} \tilde{u}$, $T_{\beta+\gamma} A$, and $T_{\beta+\gamma} f$.

Since $\| \tilde{u} \|_\infty \leqslant \nu_0$, it is clear that $\| T_\beta T_\gamma \tilde{u} \|_\infty \leqslant \nu_0$ and $\| T_{\beta+\gamma} \tilde{u} \|_\infty \leqslant \nu_0$. Hence $T_\beta T_\gamma \tilde{u}$ is the unique optimal solution of the equation

$$\Delta u + T_\beta T_\gamma A u = T_\beta T_\gamma f, \tag{12}$$

and $T_{\beta+\gamma} \tilde{u}$ is the unique optimal solution of

$$\Delta u + T_{\beta+\gamma} A u = T_{\beta+\gamma} f. \tag{13}$$

Since the coefficients A and f are almost periodic, Eq. (12) and (13) are precisely the same, hence $T_\beta T_\gamma \tilde{u} = T_{\beta+\gamma} \tilde{u}$, i.e., $\tilde{u}$ is almost periodic.

Lemma 11 completes the proof of Theorem 2(A). The proof of Theorem 2(B) can be completed by applying the methods used in Section 3. We shall not present the details here. Q.E.D.

Remark. The full strength of the hypotheses of Theorem 2(A) was not used in Lemmas 7–10. For example, Lemma 7 would be true if we only assumed the hull $H(A, f)$ to be compact. Lemma 8 is true under the assumption that the hull $H(A, f)$ be a compact minimal set. (See [4] for definition.) Lemmas 9 and 10 use the assumption that $\inf\{|v(t)|: t \in R^m\} > 0$, together with the fact that the hull $H(A, f)$ is a compact minimal set. It follows therefore that Theorem 2 remains valid if we replace "almost periodic" with "almost automorphic."

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